Journal of Approximation Theory **113**, 142–155 (2001)
doi:10.1006/jath.2001.3617, available online at http://www.idealibrary.com on IDEAL®

Asymptotic Zero Distribution of Orthogonal Polynomials with Discontinuously Varying Recurrence Coefficients

A. B. J. Kuijlaars

Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200B, 3001 Leuven, Belgium
E-mail: arno@wis.kuleuven.ac.be

and

S. Serra Capizzano

Dipartimento di Chimica, Fisica e Matematica, Università dell'Insubria - sede di Como, Via Valleggio 11, 22100 Como, Italy
E-mail: serra@mail.dm.unipi.it

Communicated by Walter Van Assche

Received November 29, 2000; accepted in revised form June 14, 2001

The zero distribution of orthogonal polynomials $p_{n,N}$, $n=0, 1, \ldots$ generated by recurrence coefficients $a_{n,N}$ and $b_{n,N}$ depending on a parameter N has been recently considered by Kuijlaars and Van Assche under the assumption that $a_{n,N}$ and $b_{n,N}$ behave like $a(n/N)$ and $b(n/N)$, respectively, where $a(\cdot)$ and $b(\cdot)$ are continuous functions. Here, we extend this result by allowing $a(\cdot)$ and $b(\cdot)$ to be measurable functions so that the presence of possible jumps is included. The novelty is also in the sense of the mathematical tools since, instead of applying complex analysis arguments, we use recently developed results from asymptotic matrix theory due to Tyrtyshnikov, Serra Capizzano, and Tilli. © 2001 Academic Press

Key Words: orthogonal polynomials; limit distribution; matrix sequences.

1. INTRODUCTION

A three term recurrence relation of the form

$$(1) \qquad xp_n(x) = a_{n+1}\, p_{n+1}(x) + b_n\, p_n(x) + a_n\, p_{n-1}(x), \qquad p_0 \equiv 1, \quad p_{-1} \equiv 0,$$

where $a_n > 0$ and $b_n \in \mathbb{R}$ generates by Favard's theorem a sequence of polynomials orthonormal with respect to a probability measure on the real line. It is well known that the zeros of the orthogonal polynomials p_n are

0021-9045/01 $35.00

real and simple, and belong to the convex hull of the support of the orthogonality measure. The asymptotic distribution of the zeros of the sequence $\{p_n\}$ of orthogonal polynomials has been studied extensively, see e.g. [9, 20] and the references cited therein. We say that the Borel measure μ on $\mathbb{R}$ is the asymptotic zero distribution of $\{p_n\}$ if the limit relation

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F(x_{j,n}) = \int F(x)\, d\mu(x),$$

where $x_{j,n}$, $j=1, \ldots, n$ are the zeros of p_n, holds for $F \in \mathscr{C}_0$, where $\mathscr{C}_0$ denotes the space of all continuous functions on $\mathbb{R}$ with bounded support.

For the case where the sequence $\{a_n\}$ converges to a limit $a>0$ and the sequence $\{b_n\}$ converges to $b \in \mathbb{R}$, Nevai [9, Theorem 5.3] proved that the asymptotic zero distribution is equal to the arcsine measure of the interval $[b-2a, b+2a]$, i.e., the measure $\omega(a, b)$ with density

$$\frac{d\omega(a,b)}{dx} = \begin{cases} \dfrac{1}{\pi\sqrt{(b+2a-x)(x-b+2a)}} & \text{if} \quad x \in (b-2a, b+2a), \\ 0 & \text{otherwise}. \end{cases} \tag{2}$$

This result may be obtained in a number of ways. One way is to observe that the zeros of p_n are the eigenvalues of the Hermitian tridiagonal Jacobi matrix J_n defined as

$$J_n = \begin{bmatrix} b_0 & a_1 & & & \\ a_1 & b_1 & a_2 & & \\ & a_2 & \ddots & \ddots & \\ & & \ddots & \ddots & a_{n-1} \\ & & & a_{n-1} & b_{n-1} \end{bmatrix}.$$

If the limits $a = \lim a_n$ and $b = \lim b_n$ exist, then the Jacobi matrix J_n can be seen as a perturbation of the nth section of the infinite Toeplitz matrix generated by $b + 2a \cos x$ (see [19] for more details). More precisely, for any $\varepsilon > 0$, we have that

$$J_n = T_n(b + 2a \cos x) + R_n(\varepsilon) + \Delta_n(\varepsilon), \tag{3}$$

where $T_n(f)$ is the nth section of the infinite Toeplitz matrix generated by f, and where

$$\operatorname{rank} R_n(\varepsilon) \leqslant C(\varepsilon), \tag{4}$$

with a constant $C(\varepsilon)$ independent of n, and

$$\|\Delta_n(\varepsilon)\| \leqslant \varepsilon. \tag{5}$$

The norm used in (5) is the spectral norm, i.e., the largest singular value. Tyrtyshnikov (see [18, Theorem 3.1]) studied such low-rank/small-norm perturbations and he proved that (3)–(5) imply that the matrix sequences $\{J_n\}$ and $\{T_n(b+2a\cos x)\}$ have the same asymptotic eigenvalue distribution. The asymptotic eigenvalue distribution of a Toeplitz sequence $\{T_n(f)\}$ generated by a real-valued, bounded f is well known from Szegő's work [4] to be the measure μ satisfying

$$\int F(x)\,d\mu(x) = \frac{1}{2\pi}\int_0^{2\pi} F(f(x))\,dx \tag{6}$$

for every $F \in \mathscr{C}_0$. For $f(x) = b+2a\cos x$, this yields the measure $\mu = \omega(a, b)$ as given by (2).

A further step is to consider a family of recurrence relations like (1) where the coefficients $a_n = a_{n,N}$ and $b_n = b_{n,N}$ depend on a parameter N. With the help of complex analysis and potential theory, Kuijlaars and Van Assche [6] proved the following result.

THEOREM 1.1 [6]. *Let for each $N \in \mathbb{N}$, two sequences $\{a_{n,N}\}_n$, $a_{n,N} > 0$ and $\{b_{n,N}\}_n$ of recurrence coefficients be given, together with orthogonal polynomials $p_{n,N}$ generated by the recurrence*

$$x p_{n,N}(x) = a_{n+1,N}\, p_{n+1,N}(x) + b_{n,N}\, p_{n,N}(x) + a_{n,N}\, p_{n-1,N}(x), \tag{7}$$

and the initial conditions $p_{0,N} \equiv 1$ and $p_{-1,N} \equiv 0$. Suppose that there exist two continuous functions a: $(0, \infty) \to [0, \infty)$, b: $(0, \infty) \to \mathbb{R}$, such that for every $t > 0$,

$$\lim_{\substack{n, N \to \infty \\ n/N \to t}} a_{n,N} = a(t), \qquad \textit{and} \qquad \lim_{\substack{n, N \to \infty \\ n/N \to t}} b_{n,N} = b(t). \tag{8}$$

Then we have for every $t > 0$ and for every $F \in \mathscr{C}_0$

$$\lim_{\substack{n, N \to \infty \\ n/N \to t}} \frac{1}{n} \sum_{j=1}^{n} F(x_{j,n,N}) = \int F(x)\,d\mu_t(x), \tag{9}$$

where $x_{j,n,N}$, $j = 1, \ldots, n$, are the zeros of $p_{n,N}$,

$$\mu_t = \frac{1}{t}\int_0^t \omega(a)(s), b(s))\,ds, \tag{10}$$

and $\omega(a, b)$ *is the arcsine measure of the interval* $[b-2a, b+2a]$ *as defined in* (2) *if* $a>0$, *and* $\omega(a, b)$ *is the unit Dirac measure in* b *if* $a=0$.

The limits in (8) mean that for every choice of sequences $\{n_j\}$ and $\{N_j\}$ such that $n_j, N_j \to \infty$, and $n_j/N_j \to t$ as $j \to \infty$ one has

$$\lim_{j\to\infty} a_{n_j, N_j} = a(t), \qquad \text{and} \qquad \lim_{j\to\infty} b_{n_j, N_j} = b(t).$$

We use $\lim_{\substack{n, N\to\infty \\ n/N\to t}}$ in this sense also in (9) and in the rest of the paper.

See [21] for an extension of Theorem 1.1 to asymptotically periodic recurrence coefficients.

Note that (9)–(10) may also be written as

$$\lim_{\substack{n, N\to\infty \\ n/N\to t}} \frac{1}{n}\sum_{j=1}^{n} F(x_{j,n,N}) = \frac{1}{\pi t}\int_0^t \int_0^\pi F(b(s)+2a(s)\cos x)\, dx\, ds. \tag{11}$$

It is the aim of this paper to study the asymptotic zero distribution of orthogonal polynomials with varying recurrence coefficients in the light of recent advances in asymptotic matrix analysis, starting with the work of Tyrtyshnikov already referred to above, and continued by Tilli and Serra Capizzano. In fact, Theorem 1.1 can be deduced from the work of Tilli [17] on locally Toeplitz sequences. Intuitively, a sequence of matrices $\{A_n\}$ is locally Toeplitz if for large n, and for k small with respect to n, the matrix entries $(A_n)_{i,j}$ and $(A_n)_{i+k,j+k}$ are close to each other. Then on a small scale, the matrix is almost a Toeplitz matrix. Globally the matrix is not Toeplitz, but on each of the diagonals of A_n the entries change only gradually. Theorem 1.1 may be viewed in this way. Indeed, the zeros of $p_{n,N}$ are eigenvalues of the Jacobi matrix

$$J_{n,N} = \begin{bmatrix} b_{0,N} & a_{1,N} & & & \\ a_{1,N} & b_{1,N} & a_{2,N} & & \\ & a_{2,N} & \ddots & \ddots & \\ & & \ddots & \ddots & a_{n-1,N} \\ & & & a_{n-1,N} & b_{n-1,N} \end{bmatrix} \tag{12}$$

and if n and N are large with $n \approx tN$, then, under the assumptions of the theorem, the entries vary gradually along the three diagonals.

These considerations led Tilli to the definition of locally Toeplitz sequence [17, Definition 1.3]. It is maybe a bit unfortunate (see [14] for a slightly different notion) that the matrices (12) do not form a locally

Toeplitz sequence as $n, N \to \infty$ such that $n/N \to t$, but instead they are the sum of two locally Toeplitz sequences with $J_{n,N} = B_{n,N} + A_{n,N}$,

$$B_{n,N} = \begin{bmatrix} b_{0,N} & 0 & & \\ 0 & b_{1,N} & & \\ & & \ddots & 0 \\ & & 0 & b_{n-1,N} \end{bmatrix}$$

and

$$A_{n,N} = \begin{bmatrix} 0 & a_{1,N} & & \\ a_{1,N} & 0 & \ddots & \\ & \ddots & \ddots & a_{n-1,N} \\ & & a_{n-1,N} & 0 \end{bmatrix}.$$

Tilli [17, Section 5] showed that the sequence $\{B_{n,N}\}$ with $n, N \to \infty$ and $n/N \to t$ is locally Toeplitz with respect to $(b_t, 1)$, and the sequence $\{A_{n,N}\}$ is locally Toeplitz with respect to (a_t, f), where

$$a_t(x) = a(ts), \qquad b_t(s) = b(ts), \qquad f(x) = 2 \cos x.$$

Then it follows from [17, Theorem 3.7] that the eigenvalues of $J_{n,N}$, which we denote by $x_{j,n,N}$, $j = 1, ..., n$ satisfy

$$\begin{aligned} \lim_{\substack{n, N \to \infty \\ n/N \to t}} \frac{1}{n} \sum_{j=1}^{n} F(x_{j,n,N}) &= \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} F(b_t(s) + a_t(s)\, f(x))\, dx\, ds \\ &= \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} F(b(ts) + 2a(ts) \cos x)\, dx\, ds \end{aligned}$$

for every $F \in \mathscr{C}_0$. This is clearly equivalent with (11).

2. STATEMENT OF RESULTS

In this paper we want to use recent results in matrix analysis to weaken the hypotheses on the recurrence coefficients $\{a_{n,N}\}$ and $\{b_{n,N}\}$ in Theorem 1.1. The limit relations (8) assumed by Kuijlaars and Van Assche in Theorem 1.1 express a certain uniform convergence and force the functions $a(t)$ and $b(t)$ to be continuous. Here we present a generalization where we only assume $a(t)$ and $b(t)$ to be measurable functions.

Our main result is the following. As usual, $[x]$ denotes the largest integer less than or equal to x.

THEOREM 2.1. *Let for each $N \in \mathbb{N}$, two sequences $\{a_{n,N}\}_n$, $a_{n,N} > 0$ and $\{b_{n,N}\}_n$ of recurrence coefficients be given, together with orthogonal polynomials $p_{n,N}$ generated by the recurrence*

$$xp_{n,N}(x) = a_{n+1,N}\, p_{n+1,N}(x) + b_{n,N}\, p_{n,N}(x) + a_{n,N}\, p_{n-1,N}(x), \qquad n \geqslant 0,$$

and the initial conditions $p_{0,N} \equiv 1$ and $p_{-1,N} \equiv 0$. Suppose that there exist two measurable functions $a\colon [0,\infty) \to [0,\infty)$ and $b\colon [0,\infty) \to \mathbb{R}$ such that for every $t > 0$ and every $\varepsilon > 0$,

$$\lim_{N\to\infty} \operatorname{meas}\{s \in [0,t] \mid |a_{[sN],N} - a(s)| \geqslant \varepsilon\} = 0, \tag{13}$$

and

$$\lim_{N\to\infty} \operatorname{meas}\{s \in [0,t] \mid |b_{[sN],N} - b(s)| \geqslant \varepsilon\} = 0. \tag{14}$$

Then we have for every $t > 0$ and every $F \in \mathscr{C}_0$,

$$\lim_{\substack{n,N\to\infty\\ n/N\to t}} \frac{1}{n} \sum_{j=1}^{n} F(x_{j,n,N}) = \frac{1}{2\pi} \int_0^t \int_0^\pi F(b(s) + 2a(s)\cos x)\, dx\, ds,$$

where $x_{j,n,N}$, $j = 1, \ldots, n$, are the zeros of $p_{n,N}$.

The assumptions (13)–(14) are equivalent to saying that the functions $s \mapsto a_{[sN],N}$ and $s \mapsto b_{[sN],N}$ *converge in measure* on every bounded interval to the functions a and b, respectively. This type of convergence is considerably more general than the convergence implied by (8).

Of particular interest is the situation where the recurrence coefficients $a_{n,N}$ and $b_{n,N}$ are exact samples of two functions a and b, i.e.,

$$a_{n,N} = a(n/N), \qquad \text{and} \qquad b_{n,N} = b(n/N). \tag{15}$$

This situation arises for example in the analysis of a continuum limit of the Toda lattice [1, 2, 5]. Also the discretization of a one-dimensional boundary value problem

$$\begin{cases} -\dfrac{d}{dx}\left(p(x)\dfrac{d}{dx}u(x)\right) + q(x)\,u(x) = f(x), & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

on a uniformly spaced grid using centered finite differences leads to a linear system with tridiagonal matrix (12) having entries of the form (15) for certain functions a and b related to p and q, cf. [10, 13, 15, 17].

The appropriate condition on the functions a and b when the coefficients $a_{n,N}$ and $b_{n,N}$ satisfy (15) is now stated in terms of Riemann integrability. As a consequence of Theorem 2.1 we have:

COROLLARY 2.2. *Let for each* $N \in \mathbb{N}$, *the sequences* $\{a_{n,N}\}_n$, $a_{n,N} > 0$ *and* $\{b_{n,N}\}_n$ *of recurrence coefficients be given by* (15) *where the functions* a: $[0, \infty) \to (0, \infty)$ *and* b: $[0, \infty) \to \mathbb{R}$ *satisfy for every* $t > 0$,

(a) *for every* $\delta > 0$, *there is* $M > 0$ *such that the set*

$$\{s \in [0, t] \mid a(s) \geqslant M \text{ or } |b(s)| \geqslant M\}$$

is contained in a finite union of intervals of total length $\leqslant \delta$.

(b) *for every* $M > 0$, *the cut-off functions*

$$s \mapsto \min(a(s), M) \qquad \text{and} \qquad s \mapsto \min(\max(b(s), -M), M) \tag{16}$$

are Riemann integrable on the interval $[0, t]$.

Let the orthogonal polynomials $p_{n,N}$ *be generated by the recurrence*

$$xp_{n,N}(x) = a_{n+1,N}\, p_{n+1,N}(x) + b_{n,N}\, p_{n,N}(x) + a_{n,N}\, p_{n-1,N}(x), \qquad n \geqslant 0,$$

with $p_{0,N} \equiv 1$ *and* $p_{-1,N} \equiv 0$. *Then we have for every* $t > 0$ *and every* $F \in \mathscr{C}_0$,

$$\lim_{\substack{n, N \to \infty \\ n/N \to t}} \frac{1}{n} \sum_{j=1}^{n} F(x_{j,n,N}) = \frac{1}{\pi t} \int_0^t \int_0^\pi F(b(s) + 2a(s) \cos x)\, dx\, ds,$$

where $x_{j,n,N}$, $j = 1, \ldots, n$ *are the zeros of* $p_{n,N}$.

Proof. The functions a and b are measurable.

Let $\varepsilon > 0$ and $\delta > 0$. By assumption (a) there is an $M > 0$ and a finite union of intervals $I_\delta \subset [0, t]$ of total length at most δ such that $a(s) \leqslant M$ for $s \in [0, t] \setminus I_\delta$. By assumption (b) the function $f(s) := \min(a(s), M)$ is Riemann integrable on $[0, t]$. The Riemann integrability implies that there exists an N_0 such that for every $N \geqslant N_0$ one has

$$\frac{1}{N} \sum_{k=0}^{[tN]} \Big(\sup_{s \in [k/N, (k+1)/N]} f(s) - \inf_{s \in [k/N, (k+1)/N]} f(s) \Big) < \varepsilon\delta. \tag{17}$$

Let $N \geqslant N_0$. It follows from (17) that the number of $k \leqslant [tN]$ for which

$$\sup_{s \in [k/N, (k+1)/N]} |f(k/N) - f(s)| \geqslant \varepsilon,$$

is less than δN. Since I_δ has total length at most δ, there are less than $\delta N+2\ell$ values of $k \leqslant [tN]$ for which $[k/N, (k+1)/N] \cap I_\delta \neq \emptyset$. Here ℓ is the number of intervals of I_δ. By definition of f and I_δ this means that there are less than $\delta N+2\ell$ values of $k \leqslant [tN]$ for which $f(s) \neq a(s)$ for some $s \in [k/N, (k+1)/N]$. Hence there are less than $2\delta N+2\ell$ values of $k \leqslant [tN]$ for which

$$\sup_{s \in [k/N, (k+1)/N]} |a(k/N)-a(s)| \geqslant \varepsilon.$$

Since $a_{k,N} = a(k/N)$ this gives

$$\operatorname{meas}\{s \in [0, t] \mid |a_{[sN], N} - a(s)| \geqslant \varepsilon\} \leqslant \frac{2\delta N+2\ell}{N}.$$

Letting first $N \to \infty$, and then $\delta \to 0+$, we find that (13) holds. In a similar way we prove (14), and the corollary follows from Theorem 2.1. ∎

As a second application we apply Theorem 2.1 to the case of a single sequence of orthogonal polynomials. The following extends results of Nevai [9], Geronimo, Harrell and Van Assche [3], and Mercer [7, 8].

Corollary 2.3. *Let $\{a_n\}_n$, $a_n > 0$, and $\{b_n\}_n$ be two sequences, generating the orthogonal polynomials p_n by the recurrence*

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x).$$

Let $a \geqslant 0$ and $b \in \mathbb{R}$ be constants such that for every $\varepsilon > 0$,

$$\#\{k \leqslant n \mid |a_k - a| \geqslant \varepsilon\} = o(n) \qquad as \quad n \to \infty, \tag{18}$$

and

$$\#\{k \leqslant n \mid |b_k - b| \geqslant \varepsilon\} = o(n) \qquad as \quad n \to \infty. \tag{19}$$

Then we have for every $F \in \mathscr{C}_0$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} F(x_{j,n}) = \frac{1}{\pi} \int_0^\pi F(b+2a\cos x)\, dx, \tag{20}$$

where $x_{j,n}$, $j = 1, \ldots, n$ are the zeros of p_n.

Proof. We write $a_{n,N} = a_n$ and $b_{n,N} = b_n$, for every $N \in \mathbb{N}$. Then for every $\varepsilon > 0$,

$$\operatorname{meas}\{s \in [0, 1] \mid |a_{[sN], N} - a| \geqslant \varepsilon\} = (1/N) \cdot \#\{k < N \mid |a_k - a| \geqslant \varepsilon\}$$

tends to 0 as $N \to \infty$ because of (18). Thus (13) holds with $t = 1$. Similarly (19) implies that (14) holds with $t = 1$. Taking $n = N$ and the constant functions $a(s) \equiv a$, $b(s) \equiv b$ in Theorem 2.1, we obtain (20). ∎

3. PROOF OF THEOREM 2.1

Our main tool is an extension of results of Tyrtyshnikov [18, Theorem 3.1] and Tilli [17, Proposition 2.7] due to Serra Capizzano [12, Proposition 2.3]. For an $n \times n$ Hermitian matrix A, we denote by $\lambda_1(A), \ldots, \lambda_n(A)$ its eigenvalues numbered in nondecreasing order.

THEOREM 3.1. *Let $\{A_n\}_n$ be a sequence of Hermitian matrices with A_n of order n. Suppose for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$, such that for every $n \geqslant n_\varepsilon$, there is a splitting*

$$A_n = B_n(\varepsilon) + R_n(\varepsilon) + \Delta_n(\varepsilon),$$

where $B_n(\varepsilon)$, $R_n(\varepsilon)$, $\Delta_n(\varepsilon)$ are Hermitian matrices such that for $n \geqslant n_\varepsilon$,

$$\text{rank } R_n \leqslant C_1(\varepsilon)\, n \qquad \text{and} \qquad \|\Delta_n(\varepsilon)\| \leqslant C_2(\varepsilon), \tag{21}$$

where $C_1(\varepsilon)$ and $C_2(\varepsilon)$ are positive constants independent of n, such that

$$\lim_{\varepsilon \to 0+} C_1(\varepsilon) = 0, \qquad \text{and} \qquad \lim_{\varepsilon \to 0+} C_2(\varepsilon) = 0. \tag{22}$$

Suppose that for every $\varepsilon > 0$ and $F \in \mathscr{C}_0$ the limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} F(\lambda_j(B_n(\varepsilon))) = \Phi_\varepsilon(F)$$

exists and that in addition for every $F \in \mathscr{C}_0$, the limit

$$\lim_{\varepsilon \to 0+} \Phi_\varepsilon(F) = \Phi(F)$$

exists. Then for every $F \in \mathscr{C}_0$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} F(\lambda_j(A_n)) = \Phi(F).$$

Proof. See [12, Proposition 2.3]. ∎

Proof of Theorem 2.1. Fix $t>0$ and let $\varepsilon>0$. By Lusin's theorem (see e.g. [11]), there exist continuous functions a_ε and b_ε on $[0,\infty)$ such that

$$\operatorname{meas}\{s\in[0,t] \mid a_\varepsilon(s)\neq a(s)\}\leqslant\varepsilon, \tag{23}$$

and

$$\operatorname{meas}\{s\in[0,t] \mid b_\varepsilon(s)\neq b(s)\}\leqslant\varepsilon. \tag{24}$$

Here meas denotes the Lebesgue measure.

Let $n, N\in\mathbb{N}$ be large but fixed (for the moment). Let $J_{n,N}$ be the $n\times n$ matrix built out of the recurrence coefficients $a_{k,N}$ and $b_{k,N}$ as in (12), and let $J_{n,N}(\varepsilon)$ be the $n\times n$ tridiagonal matrix obtained from a_ε and b_ε by sampling at spacing $1/N$. Thus

$$J_{n,N}(\varepsilon)=\begin{bmatrix} b_\varepsilon(0/N) & a_\varepsilon(1/N) & & & \\ a_\varepsilon(1/N) & b_\varepsilon(1/N) & a_\varepsilon(2/N) & & \\ & a_\varepsilon(2/N) & \ddots & \ddots & \\ & & \ddots & \ddots & a_\varepsilon((n-1)/N) \\ & & & a_\varepsilon((n-1)/N) & b_\varepsilon((n-1)/N) \end{bmatrix}.$$

The difference $J_{n,N}-J_{n,N}(\varepsilon)$ has entries $b_{n,N}-b_\varepsilon(k/N)$, $k=0,\dots,n-1$, on the diagonal, and $a_{k,N}-a_\varepsilon(k/N)$ on the first sub and superdiagonals. We let

$$J_{n,N}-J_{n,N}(\varepsilon)=R_{n,N}(\varepsilon)+\Delta_{n,N}(\varepsilon), \tag{25}$$

where $\Delta_{n,N}(\varepsilon)$ has the entries of $J_{n,N}-J_{n,N}(\varepsilon)$ that are in modulus smaller than 2ε, while $R_{n,N}(\varepsilon)$ has those entries that are in modulus at least 2ε. The other entries of $\Delta_{n,N}(\varepsilon)$ and $R_{n,N}(\varepsilon)$ are zero.

Since $\Delta_{n,N}(\varepsilon)$ is a tridiagonal matrix whose entries are all smaller than 2ε, we have for its spectral norm,

$$\|\Delta_{n,N}(\varepsilon)\|<6\varepsilon. \tag{26}$$

To estimate the rank of $R_{n,N}(\varepsilon)$ we are going to find an upper bound for the number of its non-zero entries. We concentrate on the diagonal entries.

Because of assumption (14) there is a measurable set E_1 with meas $E_1<\varepsilon$, and an $N_0\in\mathbb{N}$ such that for every $N\geqslant N_0$, we have

$$|b_{[sN],N}-b(s)|<\varepsilon, \qquad \text{for} \quad s\in[0,t]\setminus E_1.$$

Then by (24) there is a measurable set E_2 with meas $E_2<2\varepsilon$ such that

$$|b_{[sN],N}-b_\varepsilon(s)|<\varepsilon \qquad \text{for} \quad s\in[0,t]\setminus E_2. \tag{27}$$

Define

(28)
$$K(n, N, \varepsilon) = \left\{ k \in \{0, 1, ..., n-1\} \;\middle|\; \left[\frac{k}{N}, \frac{k+1}{N}\right) \cap ([0, t] \setminus E_2) \neq \varnothing \right\}.$$

It follows from (27) and the fact that meas $E_2 \leqslant 2\varepsilon$ that

(29)
$$\# K(n, N, \varepsilon) \geqslant \min(n, tN) - 2\varepsilon N.$$

Since b_ε is continuous, it is uniformly continuous on $[0, t]$. Therefore there is $\delta > 0$ such that for every $s_1, s_2 \in [0, t]$ with $|s_1 - s_2| < \delta$, we have $|b_\varepsilon(s_1) - b_\varepsilon(s_2)| < \varepsilon$.

Let now $N \geqslant N_1 := \max(N_0, 1/\delta)$. If $k \in K(n, N, \varepsilon)$ then because of (27) and (28) there exists $s \in [k/N, (k+1)/N)$ such that $|b_{k,N} - b_\varepsilon(s)| < \varepsilon$. Then $|k/N - s| < 1/N < \delta$ so that $|b_\varepsilon(s) - b_\varepsilon(k/N)| < \varepsilon$. Hence

(30)
$$|b_{k,N} - b_\varepsilon(k/N)| < 2\varepsilon, \qquad \text{for} \quad k \in K(n, N, \varepsilon).$$

It follows from the construction of $R_{n,N}(\varepsilon)$ and (30) that the $(k+1, k+1)$-entry of $R_{n,N}(\varepsilon)$ is zero if $k \in K(n, N, \varepsilon)$. Then (29) implies that the number of non-zero diagonal entries of $R_{n,N}(\varepsilon)$ is at most

$$n - \min(n, tN) + 2\varepsilon N = \max(0, n - tN) + 2\varepsilon N.$$

This holds provided $N \geqslant N_1$.

In a similar way, we show that there exists N_2 such that the total number of non-zero entries on the sub and superdiagonals of $R_{n,N}(\varepsilon)$ is at most $2(\max(0, n - tN) + 2\varepsilon N)$ for $N \geqslant N_2$. Hence the total number of non-zero entries of $R_{n,N}(\varepsilon)$ is at most $3 \max(0, n - tN) + 6\varepsilon N$ if $N \geqslant \max(N_1, N_2)$ and therefore,

(31)
$$\operatorname{rank} R_{n,N}(\varepsilon) \leqslant 3 \max(0, n - tN) + 6\varepsilon N, \qquad \text{for} \quad N \geqslant \max(N_1, n_2).$$

All of this holds for fixed n and N with N large enough. We now let $n, N \to \infty$ in such a way that $n/N \to t$. Then $\max(0, n - tN) = o(n)$ as $n \to \infty$. It follows from (31) that for n (and corresponding N) large enough,

(32)
$$\operatorname{rank} R_{n,N}(\varepsilon) \leqslant 10 t^{-1} \varepsilon n.$$

Then (26) and (32) imply that the rank and norm conditions (21) and (22) of Theorem 3.1 are satisfied. In addition, we have by Theorem 1.1 (see also [17]), since a_ε and b_ε are continuous,

$$\lim_{\substack{n, N \to \infty \\ n/N \to t}} \frac{1}{N} \sum_{j=1}^{n} F(\lambda_j(J_{n,N}(\varepsilon))) = \int F(x)\, d\left[\frac{1}{t}\int_0^t \omega(a_\varepsilon(s), b_\varepsilon(s))\, ds\right]$$

$$= \frac{1}{\pi t}\int_0^t \int_0^\pi F(b_\varepsilon(s) + 2a_\varepsilon(s)\cos x)\, dx\, ds.$$

for every $F \in \mathscr{C}_0$.

From (23) and (24) it next follows that for every fixed x, the functions $F(b(s) + 2a(s)\cos x)$ and $F(b_\varepsilon(s) + 2a(s)\cos x)$ are equal for $s \in [0, t]$ except for a set of measure less than 2ε. This easily implies that for $F \in \mathscr{C}_0$,

$$\left|\frac{1}{\pi t}\int_0^t \int_0^\pi F(b_\varepsilon(s) + 2a_\varepsilon(s)\cos x)\, dx\, ds - \frac{1}{\pi t}\int_0^t \int_0^\pi F(b(s) + 2a(s)\cos x)\, dx\, ds\right|$$

$$\leqslant \frac{1}{\pi t}\int_0^t \int_0^\pi |F(b_\varepsilon(s) + 2a_\varepsilon(s)\cos x) - F(b(s) + 2a(s)\cos x)|\, dx\, ds$$

$$\leqslant \frac{4\, \|F\|_\infty}{t}\, \varepsilon.$$

Therefore

$$\lim_{\varepsilon \to 0+} \frac{1}{\pi t}\int_0^t \int_0^\pi F(b_\varepsilon(s) + 2a_\varepsilon(s)\cos x)\, dx\, ds$$

$$= \frac{1}{\pi t}\int_0^t \int_0^\pi F(b(s) + 2a(s)\cos x)\, dx\, ds.$$

Hence all the conditions of Theorem 3.1 are satisfied and we see that

$$\lim_{\substack{n, N \to \infty \\ n/N \to t}} \frac{1}{N} \sum_{j=1}^{n} F(\lambda_j(J_{n,N})) = \frac{1}{\pi t}\int_0^t \int_0^\pi F(b(s) + 2a(s)\cos x)\, dx\, ds.$$

Then Theorem 2.1 follows, since the zeros of $p_{n,N}$ are equal to the eigenvalues of $J_{n,N}$. ∎

We remark that, in the case where $a(t)$ and $b(t)$ are Riemann integrable on any bounded interval, Theorem 2.1 proves that the sequence $\{J_n\}$ is a Generalized Locally Toeplitz (GLT) sequence in the sense of [16] with respect to the kernel $b_t(x)+2a_t(x)\cos(y)$. Conversely, our result suggested a generalization of the definition of GLT sequences which takes into account measurable kernels and replaces the L^1 convergence with the weaker convergence in measure. For the analysis of this extended notion we refer the reader to [14] and, more specifically, to Definition 2.3, Theorem 4.5 and Remark 6.3 in [14].

REFERENCES

1. A. Bloch, F. Golse, T. Paul, and A. Uribe, Dispersionless Toda and Toeplitz operators, preprint.
2. P. Deift and K. T-R McLaughlin, A continuum limit of the Toda lattice, *Mem. Amer. Math. Soc.* **131** (1998), 624.
3. J. S. Geronimo, E. M. Harrell II, and W. Van Assche, On the asymptotic distribution of eigenvalues of banded matrices, *Constr. Approx.* **4** (1988), 403–417.
4. U. Grenander and G. Szegő, "Toeplitz Forms and Their Applications," 2nd ed., Chelsea, New York, 1984.
5. A. B. J. Kuijlaars, On the finite-gap ansatz in the continuum limit of the Toda lattice, *Duke Math. J.* **104** (2000), 433–462.
6. A. B. J. Kuijlaars and W. Van Assche, The asymptotic zero distribution of orthogonal polynomials with varying weights, *J. Approx. Theory* **99** (1999), 167–197.
7. A. McD. Mercer, The distribution of the zeros of certain orthogonal polynomials, *J. Approx. Theory* **60** (1990), 183–187.
8. A. McD. Mercer, A note on the zero distribution of orthogonal polynomials, *J. Approx. Theory* **64** (1991), 238–242.
9. P. G. Nevai, Orthogonal polynomials, *Mem. Amer. Math. Soc.* **18** (1979), 213.
10. S. Parter, On the eigenvalues of certain generalizations of Toeplitz matrices, *Arch. Rational Math. Mech.* **3** (1962), 244–257.
11. W. Rudin, "Real and Complex Analysis," 3rd ed., McGraw–Hill, New York, 1986.
12. S. Serra Capizzano, Distribution results on the algebra generated by Toeplitz sequences: a finite dimensional approach, *Linear Algebra Appl.* **328** (2001), 121–130.
13. S. Serra Capizzano, Spectral behaviour of matrix sequences and discretized boundary value problems, *Linear Algebra Appl.*, in press.
14. S. Serra Capizzano, Generalized locally Toeplitz sequences: spectral analysis and applications to discretized partial differential equations, *Linear Algebra Appl.*, to appear.
15. S. Serra Capizzano and C. Tablino Possio, Spectral and structural analysis of high precision finite difference matrices for elliptic operators, *Linear Algebra Appl.* **293** (1999), 85–131.
16. S. Serra Capizzano and P. Tilli, "From Partial Differential Equations to Generalized Locally Toeplitz Sequences," TR-99–12, Dept. Informatica, Univ of Pisa, 1999, available at http://www.di.unipi.it/ricerca/TR/tr1999.html.
17. P. Tilli, Locally Toeplitz sequences: spectral properties and applications, *Linear Algebra Appl.* **278** (1998), 91–120.
18. E. E. Tyrtyshnikov, A unifying approach to some old and new theorems on distribution and clustering, *Linear Algebra Appl.* **232** (1996), 1–43.
19. E. E. Tyrtyshnikov, A matrix view on the root distribution for orthogonal polynomials, *in* "Structured Matrices: Recent Developments in Theory and Computation" (D. Bini, E. Tyrtyshnikov, and P. Yalamov, Eds.), Nova Science, New York, 2001.

20. W. Van Assche, "Asymptotics for Orthogonal Polynomials," Lecture Notes in Mathematics, Vol. 1265, Springer-Verlag, Berlin, 1987.
21. W. Van Assche, Zero distribution of orthogonal polynomials with asymptotically periodic varying recurrence coefficients, *in* "Self-Similar Systems" (V. B. Priezzhev and V. P. Spiridonov, Eds.), pp. 392–402, Joint Institute for Nuclear Research, Dubna, Russia, 1999.